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TRANSIENT DYNAMIC CONTACT PROBLEMS OF THE THEORY OF ELASTICITY WITH VARYING WIDTH OF THE CONTACT ZONE†

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Two transient dynamic contact problems involving the penetration of a rigid punch into an elastic half-space are considered. In the first problem the punch is wedge-shaped, in the second, a paraboloid. The problems are solved by a method developed in previous publications [1, 2]. The requirements imposed on the smoothness of the solutions of the problems lead to additional conditions, due to which the width of the contact area between the punch and the elastic half-space, which varies with time, is determined as a function of time and of the law governing the penetration of punch into the elastic half-space, which is determined from the differential equation of motion of a massive punch on the elastic half-space. © 2004 Elsevier Ltd. All rights reserved.

Solutions of mixed problems of the theory of elasticity in which the curve, where the boundary conditions are changed, varies with time, have been considered in analytical form in several publications [3, 4, etc.]. A detailed bibliography of the class of problems is given in [5].

1. FORMULATION OF THE PROBLEMS AND THEIR INTEGRAL EQUATIONS

We will consider transient dynamic contact problem of the penetration of a rigid punch into an elastic half-space ($-\infty < x < \infty$, $y \geq 0$) for the case of a wedge-shaped punch (problem 1) and a paraboloid punch (problem 2). The penetration of the punches into the half-space occurs along the y-axis ($x = 0$), which is their axis of symmetry. The initial velocity of the punches is υ_0, the mass per unit length of each in m, and the half-width of their contact area with the elastic half-space is $a(t)$, a positive-valued function of the time t. There are no forces of friction and adhesion in the contact area of the punches with the elastic medium. The shape of the punches and the law governing their penetration into the elastic medium are given by the function $g(x, t)$ ($t > 0$, $|x| \leq a(t)$). In the case of a wedge-shaped punch (problem 1) this function has the form

$$g(x, t) = \varepsilon(t) - \theta_1|x|, \quad \theta_1 = \operatorname{ctg}\alpha \tag{1.1}$$

where 2α is the angle of the wedge and $\varepsilon(t)$ is the law governing its penetration into the elastic medium. In the case of a paraboloid punch (problem 2) the function is

$$g(x, t) = \varepsilon(t) - \theta_2 x^2 \tag{1.2}$$

where θ_2 is a parameter characterizing the flatness (slope) of the parabola-shaped punch, which has the dimensions of m^{-1}.

At the initial instant of time the elastic half-space is at rest and therefore the displacements of the elastic medium $u = u(x, y, t)$ and $\upsilon = \upsilon(x, y, t)$ and their velocities vanish at $t = 0$.

The boundary conditions of the two problems in the standard notation of the theory of elasticity [6, 7] have the form ($t > 0$)

$$\upsilon(x, 0, t) = g(x, t), \quad |x| \leq a \tag{1.3}$$

$$\sigma_{yy}(x, 0, t) = 0, \quad a < |x| < \infty; \quad \sigma_{xy}(x, 0, t) = 0, \quad |x| < \infty \tag{1.4}$$

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where σ_{yy}, σ_{xy} are the normal and shear stresses. At infinity (as $\sqrt{x^2+y^2}\to\infty$) the stresses and displacements in the elastic half-space vanish.

By successive application of Laplace transformations (with respect to time x) with parameter p and Fourier transformations (with respect to the x coordinate) [8] to the differential equations of the theory of elasticity [6, 7] and to boundary conditions (1.3) and (1.4), taking into account the conditions at infinity and zero initial data, problems 1 and 2 can be reduced to the solution of an integral equation of the first kind in non-dimensional form [1, 2]

$$\int_{-1}^{1}\varphi^L(\xi,p)k\left(\frac{\xi-x}{\Lambda}\right)d\xi = 2\pi f^L(x,p),\quad |x|\le 1 \tag{1.5}$$

$$k(t) = \int_{\Gamma}K(u)e^{iut}du,\quad K(u) = 2(1-\beta^2)\sigma_2R^{-1}(u) \tag{1.6}$$

$$\begin{gathered}R(u) = (2u^2+1)^2-4u^2\sigma_1\sigma_2;\quad \sigma_1 = \sqrt{u^2+1},\quad \sigma_2 = \sqrt{u^2+\beta^2}\\ \Lambda = \frac{c_2}{pa},\quad \beta = \frac{c_2}{c_1},\quad c_1 = \sqrt{\frac{\lambda+2\mu}{\rho}},\quad c_2 = \sqrt{\frac{\mu}{\rho}}\end{gathered} \tag{1.7}$$

where $\varphi^L(x,p)$ is the Laplace transform of $\varphi(x,t)$, which is the unknown distribution function of the contact stresses beneath the punch, $f(x,p) = \Delta g^L(x,p)$, $\Delta = 2(1-\beta^2)\mu a^{-1}$, and $g^L(x,p)$ is the Laplace transform of the function $g(x,t)$, which describes the shape of the punch and the law governing its penetration into the elastic medium (1.1), (1.2)

$$g^L(x,p) = \varepsilon^L(p)-\theta_1ap^{-1}|x|,\quad |x|\le 1\quad \text{for problem 1} \tag{1.8}$$

$$g^L(x,p) = \varepsilon^L(p)-\theta_2a^2p^{-1}x^2,\quad |x|\le 1\quad \text{for problem 2} \tag{1.9}$$

where $\varepsilon^L(p)$ is the Laplace transform of the function $\varepsilon(t)$ from (1.1) and (1.2), $a = a(t)$ is the half-width of the contact area ($a(t)\ge 0$), c_1 and c_2 are the velocities of propagation of longitudinal and transverse elastic waves of displacements and stresses, λ and μ are the Lamé coefficients and ρ is the density of the material of the elastic medium. The contour of integration Γ in the complex plane $u = \sigma+i\tau$ goes from $-\infty$ to $+\infty$ along the real axis ($\tau = 0$) at an angle $-\arg p$ to its positive direction.

2. THE SYMBOL OF THE KERNEL OF INTEGRAL EQUATION (1.5) AND ITS BASIC PROPERTIES

The function $K(u)$ (the second equality in (1.6)) – the symbol of the kernel of Eq. (1.5) – is an even function and real-valued on the real axis of the complex plane $u = \sigma+i\tau$. Its asymptotic behaviour at zero and at infinity are defined by the following relations

$$K(u) = |u|^{-1}+O(|u|^{-3})\quad \text{as}\quad |u|\to\infty \tag{2.1}$$

$$\begin{gathered}K(u) = K(0)+\frac{1}{2!}K''(0)u^2+O(u^4)\quad \text{as}\quad u\to 0\\ K(0) = 2\beta(1-\beta^2),\quad K''(0) = 2\beta^{-1}(1-9\beta^2+8\beta^3+8\beta^4-8\beta^5)\end{gathered} \tag{2.2}$$

In the complex plane $u = \sigma+i\tau$ the function $K(u)$ has four branch points $u = \pm i\beta$, $u = \pm i$ and two Rayleigh poles $u = \pm i\eta_0$, determined from the Rayleigh equation $R(iu) = 0$ [7].

For a single-valued representation of the function $K(u)$, the complex plane $u = \sigma+i\tau$ is cut from the branch point $u = i$, $u = i\beta$ to $+i\infty$ along the positive part of the imaginary axis ($\operatorname{Im}u > 0$) and from the branch points $u = -i$, $u = -i\beta$ ($\beta > 0$) to $-i\infty$ along the negative part of the imaginary axis ($\operatorname{Im}u\le 0$). In the cut complex plane $u = \sigma+i\tau$ punctured at the Rayleigh poles $u = \pm i\eta_0$ the function $K(u)$ is analytic, including the strip $|\operatorname{Im}(u)| < \beta$, ($\beta < 1 < \eta_0$).

3. THE ASYMPTOTIC SOLUTION OF EQ. (1.6)

The zeroth term of the asymptotic expansion of the solution of integral equation (1.5), $\varphi^L(x, p)$, may be constructed for small Λ (large p) according to the formula [1, 2, 9]

$$\varphi^L(x,p) = \varphi_+^L\left(\frac{1+x}{\Lambda}, p\right) + \varphi_-^L\left(\frac{1-x}{\Lambda}, p\right) - \varphi_\infty^L\left(\frac{x}{\Lambda}, p\right), \quad |x| \le 1 \tag{3.1}$$

where the functions $\varphi_\pm^L(x, p)$ and $\varphi_\infty^L(x, p)$ are given by the integral equations

$$\int_0^\infty \varphi_\pm^L(\xi, p)k(\xi - x)d\xi = 2\pi f^L(\pm\Lambda x \mp 1, p)\Lambda^{-1}, \quad 0 \le x < \infty \tag{3.2}$$

$$\int_0^\infty \varphi_\infty^L(\xi, p)k(\xi - x)d\xi = 2\pi f^L(\Lambda x, p)\Lambda^{-1}, \quad -\infty < x < \infty \tag{3.3}$$

The kernel $k(t)$ (1.6), after deformation of the contour of integration Γ into the real axis, has the form

$$k(t) = \int_{-\infty}^{\infty} K(u)e^{iut}du$$

Equations (3.2) are Wiener–Hopf integral equations on the half-axis [10], and (3.3) is a Fourier convolution equation on the axis [11].

A solution of Eq. (2.3) is found using the integral Fourier transformation, it is given by the formula

$$\varphi_\infty^L(x,p) = \frac{1}{2\pi\Lambda}\int_{-\infty}^{\infty}\frac{f^{LF}(u,p)}{K(u)}e^{-iux}du \tag{3.4}$$

where

$$f^{LF}(u,p) = 2\pi\lambda_0\delta(u) - 2\lambda_1\Lambda a u^{-2}, \quad \text{for problem } 1 \tag{3.5}$$

$$f^{LF}(u,p) = 2\pi\lambda_0\delta(u) - 2\lambda_2\Lambda^2\delta''(u), \quad \text{for problem } 2 \tag{3.6}$$

where $\lambda_0 = 2(1-\beta^2)\mu a^{-1}\varepsilon^L(p)$, $\lambda_1 = -2(1-\beta^2)\mu\theta_1 p^{-1}$, $\lambda_2 = -2(1-\beta^2)\mu a\theta_2 p^{-1}$ and $\delta(u)$ is the Dirac delta function in the complex plane of u (primes denote derivatives).

After computing the quadratures in (3.4), we obtain the problem 1

$$\Delta^{-1}\varphi_\infty^L(x,p) = \frac{1}{\Lambda K(0)}\left[\varepsilon^L(p) - \frac{\theta_1 a^2}{c_2}\left(\frac{2}{\pi}K(0)\int_\beta^\infty\frac{l(\xi)}{\xi^2}\exp(-|x|\xi)d\xi + |x|\right)\right]$$

$$l(u) = \begin{cases} l_1(u) - 4u^2\sigma_{10}\sigma_{20}l_0^{-1}(u), & 1 \le u < \infty \\ l_1(u), & \beta \le u < 1 \end{cases} \tag{3.7}$$

$$l_1(u) = (2u^2-1)^2 l_0^{-1}(u), \quad l_0(u) = 2(1-\beta^2)\sigma_{20}, \quad \sigma_{10} = \sqrt{u^2-1}, \quad \sigma_{20} = \sqrt{u^2-\beta^2}$$

and for problem 2

$$\Delta^{-1}\varphi_\infty^L(x,p) = \frac{1}{\Lambda K(0)}\left[\varepsilon^L(p) - \frac{\theta_2 a^3}{c_2}\Lambda\left(x^2 + \Lambda^2\frac{K''(0)}{K(0)}\right)\right] \tag{3.8}$$

The derivation of (3.7) and (3.8) used the fact that $K'(0) = 0$ since $K(u)$ is an even function.

To solve integral equation (3.2), one can use the standard solution procedure of the Wiener–Hopf method [1, 2, 10–12]. Consider Eq. (3.2) for $\varphi_+^L(x, p)$, extending it to the entire real axis as follows:

$$\int_0^\infty \varphi_+^L(\xi, p)k(\xi - x)d\xi = \begin{cases} 2\pi f^L(\Lambda x - 1, p)\Lambda^{-1}, & 0 \le x < \infty \\ 2\pi \upsilon_-^L(x, p), & -\infty < x < 0 \end{cases} \tag{3.9}$$

By $\upsilon_-^L(x, p)$ we mean the integral operator

$$\upsilon_-^L(x, p) = \frac{1}{2\pi}\int_0^\infty \varphi_+^L(\xi, p)k(\xi - x)d\xi \tag{3.10}$$

which defines the Laplace transform of the elastic vertical displacements $\upsilon(x, t)$ of the surface of the elastic medium outside the punch.

Taking applying an integral Fourier transformation to Eq. (3.9), we obtain the functional equation

$$K(u)\varphi_+^{LF}(u, p) = \Lambda^{-1} f_+^{LF}(u, p) + \upsilon_-^{LF}(u, p) \tag{3.11}$$

$$\begin{gathered} \varphi_+^{LF}(u, p) = \int_0^\infty \varphi_+^L(\xi, p)e^{iu\xi}d\xi \\ f_+^{LF}(u, p) = \int_0^\infty f^L(\Lambda\xi - 1, p)e^{iu\xi}d\xi, \quad \upsilon_-^{LF}(u, p) = \int_{-\infty}^0 \upsilon_-^L(\xi, p)e^{iu\xi}d\xi \end{gathered} \tag{3.12}$$

for the unknown Laplace–Fourier transform of the function $\varphi_+^{LF}(u, p)$, which is the Fourier transform of the unknown function $\varphi_+^L(x, p)$. The function $f_+^{LF}(u, p)$ in the case of problem 1 is given by

$$f_+^{LF}(u, p) = \Delta\Lambda^{-1}\lfloor -(\varepsilon^L(p) - \eta_1\Lambda)(iu)^{-1} + \eta_1\Lambda^2(1 - 2\exp(iu\Lambda^{-1}))(iu)^{-2}\rfloor \tag{3.13}$$

and in the case of problem 2,

$$\begin{gathered} f_+^{LF}(u, p) = \Delta\Lambda^{-1}\lfloor -(\varepsilon^L(p) - \eta_2\Lambda)(iu)^{-1} + 2\eta_2\Lambda^2(1 + \Lambda(iu)^{-1})(iu)^{-2}\rfloor \\ \eta_k = \theta_k a^{1+k} c_2^{-1}, \quad k = 1, 2 \end{gathered} \tag{3.14}$$

The functions $\varphi_+^{LF}(u, p)$ and $f_+^{LF}(u, p)$ are regular in the upper half-plane $\mathrm{Im}(u) > 0$, while $\upsilon_-^{LF}(u, p)$ is regular in the lower half-plane $\mathrm{Im}(u) < \beta$, $\beta > 0$ of the complex plane $u = \sigma + i\tau$, and the function $K(u)$ is regular in the strip $|\mathrm{Im}(u)| < \beta$. Assuming that the function $K(u)$ can be factorized [10]

$$K(u) = K_+(u)K_-(u) \tag{3.15}$$

where the functions $K_+(u)$ and $K_-(u)$ are regular in the upper half-plane ($\mathrm{Im}(u) > -\beta$) and the lower half-plane ($\mathrm{Im}(u) < \beta$), respectively, we substitute expression (3.15) into Eq. (3.11) and divide the left- and right-hand sides by $K_-(u)$. The function thus obtained

$$g(u, p) = \Lambda K_-^{-1}(u) f_+^{LF}(u, p) \tag{3.16}$$

may be expressed as the sum of two functions [10],

$$g(u, p) = g_+(u, p) + g_-(u, p) \tag{3.17}$$

where $g_+(u, p)$ is regular in the upper half-plane ($\mathrm{Im}(u) > 0$) and $g_-(u, p)$ is regular in the lower half-plane ($\mathrm{Im}(u) < \beta$) of $u = \sigma + i\tau$. In the case of problem 1 we have

$$g_+(u, p) = \Delta\sum_{k=1}^{2}\sum_{n=0}^{1} c_{kn}(p)g_{kn}^+(u) \tag{3.18}$$

$$c_{10}(p) = -(\varepsilon^L(p) - \eta_1\Lambda(1 + \Lambda\gamma'_- - 2\Lambda\gamma_p))\Lambda^{-1}$$

$$c_{11}(p) = c_{21}(p) = -2c_{20}(p) = -2\eta_1\Lambda$$

$$g^+_{k0}(u) = \frac{1}{(iu)^k K_-(0)}, \quad k = 1, 2, \quad g^+_{11}(u) = \frac{1}{2\pi}\int_{\Gamma_{0+}} \frac{\exp(i\zeta\Lambda^{-1})}{\zeta K_-(\zeta)}\frac{d\zeta}{\zeta - u}\frac{1}{iu}$$

$$g^+_{21}(u) = \frac{\exp(iu\Lambda^{-1})}{(iu)^2 K_-(0)}, \quad \gamma'_- = \frac{iK'_-(0)}{K_-(0)}, \quad \gamma_p = \frac{K_-(0)}{2\pi}\int_{\Gamma_{0+}} \frac{\exp(i\zeta\Lambda^{-1})}{(i\zeta)^2 K_-(\zeta)}d\zeta$$

and in the case of problem 2

$$g_+(u, p) = \Delta\sum_{n=1}^{3} c_n(p) g^+_{n0}(u)$$

$$\begin{aligned} c_1(p) &= -(\varepsilon^L(p) - \eta_2 a(1 + 2\Lambda^2\gamma'_- + 2\Lambda^3(\gamma''_- + \gamma'^2_-)))\Lambda^{-1} \\ c_2(p) &= 2\eta_2\Lambda(1 + 2\Lambda\gamma'_-), \quad c_3(p) = 4\eta_2\Lambda^2 \end{aligned} \tag{3.19}$$

$$g^+_{30}(u) = \frac{1}{(iu)^3 K_-(0)}; \quad \gamma''_- = \frac{K''_-(0)}{K_-(0)}$$

Formulae for $g^+_{10}(u)$ and $g^+_{20}(u)$ are given in (3.18), and the constants η_k are given in (3.13) and (3.14). The contour of integration Γ_{0+} runs in the upper half-plane along the cuts from $+i\infty$ to $i\beta$ along the imaginary axis (from the right), and from $i\beta$ to $+i\infty$ (from the left); the branches for evaluating the roots σ_1 and σ_2 are chosen so that along the sides of the cuts

$$\sqrt{-u^2 + \beta^2} = \pm i\sqrt{u^2 - \beta^2}, \quad \sqrt{-u^2 + 1} = \pm i\sqrt{u^2 - 1}$$

where the upper sign is taken for the right side and the lower one for the left. Primes denote derivatives.

As a result of representation (3.17), the functional equation becomes

$$\varphi^{LF}_+(u, p)K_+(u) - g_+(u, p) = g_-(u, p) + \upsilon^{LF}_-(u, p)K^{-1}_-(u) \tag{3.20}$$

and in view of the decrease at infinity of all the functions in (3.20) and Liouville's theorem, Eq. (3.20) implies two equalities

$$\varphi^{LF}_+(u, p)K_+(u) - g_+(u, p) = 0 \tag{3.21}$$

$$\upsilon^{LF}_-(u, p)K^{-1}_-(u) + g_-(u, p) = 0 \tag{3.22}$$

for determining $\varphi^{LF}_+(u, p)$ and $\upsilon^{LF}_-(u, p)$.

The required solution $\varphi^L_+(x, p)$ of Eq. (3.2) is determined by an inverse Fourier transformation of $\varphi^{LF}_+(u, p)$ from relation (3.21)

$$\varphi^L_+(x, p) = \frac{1}{2\pi}\int_{-\infty+ic}^{\infty+ic} \frac{g_+(u, p)}{K_+(u)}e^{-iux}du, \quad c > 0, \quad 0 \le x < \infty \tag{3.23}$$

In the case of problem 1, the function $g_+(u, p)$ is given by (3.18), and in the case of problem 2 – by (3.19).

Note that relation (3.22) may also be used to determine the second unknown function $\upsilon^{LF}_-(u, p)$ and then, reverting back from the Fourier transform to the original function, we obtain

$$\upsilon^L_-(x, p) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g_-(u, p)K_-(u)e^{-iux}du, \quad -\infty < x < 0$$

where

$$g_-(u,p) = g(u,p) - g_+(u,p)$$

with the function $g(u,p)$ given by (3.16).

The solution $\varphi_-^L(x,p)$ of the second equation of (3.2) is identical with $\varphi_+^L(x,p)$ ($\varphi_-^L(x,p) = \varphi_+^L(x,p)$) because, for both problems (1 and 2) under consideration, $f(x, t)$ are even functions of x and $F^L(+\Lambda x - 1, p) = f^L(-\Lambda x + 1, p)$ in (3.2). For the same reason also $\upsilon_+^L(x,p) = \upsilon_-^L(x,p)$.

Note that in order to compute the quadratures in formula (3.23), which determines the solution $\varphi_+^L(x,p)$ of Eq. (3.2), one has to know the functions $K_\pm(u)$. In the general case, they are given in singular quadratures [7], which complicates the analysis of the results and their numerical implementation.

4. APPROXIMATION OF THE SYMBOL OF THE KERNEL OF EQ. (1.5)

To obtain an effective solution of Eq. (3.2), we replace the function $K(u)$ – the symbol of the kernel (1.6) of Eq. (3.2) – by the approximating function $K_0(u)$ proposed in [1, 2]. We have

$$K_0(u) = \sqrt{u^2+\beta^2}(u^2+\eta_0^2)^{-1}\exp[M_n^+(u)+M_n^-(u)] \tag{4.1}$$

$$M_n^\pm(u) = \frac{1}{2}\sum_{k=0}^{n} d_k(\sqrt{\beta \pm iu} - \sqrt{1 \pm iu})^{2k+2}$$

The constants d_k are determined from the conditions of the best approximation of $K(u)$ in the complex plane $u = \sigma + i\tau$ with the cuts described in Section 2. In that domain $K_0(u)$ is a univalent analytic function. Its factorization, that is, representation $K_0(u) = K_+^0(u)K_-^0(u)$, is obtained by elementary means, and it turns out that

$$K_\pm^0(u) = \frac{\sqrt{\beta \mp iu}}{\eta_0 \mp iu}\exp[M_n^\pm(u)] \tag{4.2}$$

The basic properties of the functions $K_0(u)$ and $K_\pm^0(u)$ were indicated in [1].

From a technical point of view, approximation of the symbol of the kernel $K(u)$ (1.6) by the function (4.1) reduces to determining the approximation coefficients d_k ($k = 0, 1, 2, \ldots, n$). They may be determined by various classical methods of the theory of the approximation of functions in the complex domain [13]. Since the asymptotic behaviour of the function $K_0(u)$ at infinity (as $|u| \to \infty$) is identical with that of $K(u)$ (2.1), it will suffice to approximate $K(u)$ (1.6) by $K_0(u)$ in a circle centred at the origin. Taking the power properties of the functions $K(u)$ (2.2) and $K_0(u)$ (4.1) in the neighbourhood of zero ($u = 0$) into consideration, one can determine the constants d_k by using power-series expansions of these functions. This leads to the conditions that the functions $K(u)$ and $K_\pm(u)$ are identical with $K_0(u)$ and $K_\pm^0(u)$ and that their respective derivatives are identical at zero, which may be written as follows (the superscript j denotes the order of the derivative):

$$K^{(j)}(0) = K_0^{(j)}(0), \quad j = 0, 2, \ldots, 2m \tag{4.3}$$

$$K_\pm^{(j)}(0) = K_\pm^{0(j)}(0), \quad j = 1, 3, 5, \ldots, 2m-1 \tag{4.4}$$

where necessarily $n = 2m$, and either the upper of lower plus and minus signs are chosen in (4.4). The following properties of the function $K_\pm^0(u)$ are also taken into account

$$K_+^0(u) = K_-^0(-u); \quad K_+^{0(j)}(u) = (-1)^j K_-^{0(j)}(-u), \quad j = 1, 2, \ldots \tag{4.5}$$

To ensure the satisfaction of conditions (4.4), one has to determine the functions $K_\pm^j(0)$, which is readily done by classical means using the formula

$$K_+^{(j)}(u) = \frac{K_+(u)}{2\pi i}\int_{-\infty}^{\infty}\frac{K'(\alpha)}{K(\alpha)}\frac{d\alpha}{(\alpha-u)^j}, \quad j = 1, 2, \ldots \tag{4.6}$$

and taking into account the equality

$$K_\pm(0) = \sqrt{K(0)} \tag{4.7}$$

Since $K(u)$ is even, we have $K^{(2j-1)}(0) = 0$ ($j = 1, 2, \ldots$) and all the $K_\pm^{(2j)}(0)$ are expressed in terms of $K^{(2j)}(0)$ and $K_\pm^{(2j-1)}(0)$, for example

$$K''_\pm(0) = \frac{1}{K_\pm(0)}\left[\frac{1}{2}K''(0) + K'^2_\pm(0)\right] \tag{4.8}$$

The approximation (4.1) for $n = 0$ has been used previously [1, 2]. Solution of the problems (1 and 2) formulated here requires a more accurate approximation, since the general formulae (3.23) obtained above for the solutions contain derivatives of the functions $K(u)$ and $K_\pm(u)$ at $u = 0$. For $n = 2$ ($m = 1$) conditions (4.3) and (4.4) become

$$K(0) = K_0(0), \quad K'_+(0) = K_+^{0'}(0), \quad K''(0) = K''_0(0), \tag{4.9}$$

from which we obtain a system of linear algebraic equations for determining the constants d_0, d_1 and d_2

$$\begin{aligned} d_0 + bd_1 + b^2 d_2 &= b_1 \\ d_0 + 2bd_1 + 3b^2 d_2 &= b_2 \\ d_0 + 2b\varepsilon_1 d_1 + 3b^2\varepsilon_2 d_2 &= b_3 \end{aligned} \tag{4.10}$$

in which

$$b_1 = b^{-1}\ln(2(1-\beta^2)\eta_0^2), \quad b_2 = -(2\beta - \eta_0 + 2\eta_0\beta c_0)(b\eta_0\sqrt{\beta})^{-1}$$

$$b_3 = 4\beta\sqrt{\beta}(4\eta_0^2 - 4\eta_0^2\beta - 1)(db)^{-1}, \quad \varepsilon_1 = (\beta + 4\sqrt{\beta} + 1)d^{-1}, \quad \varepsilon_2 = (\beta + 6\sqrt{\beta} + 1)d^{-1}$$

$$d = (1+\sqrt{\beta})^2, \quad b = (1-\sqrt{\beta})^2, \quad c_0 = \frac{1}{\pi}\int_0^\infty \frac{K'(u)}{uK(u)}du$$

The derivation of relations (4.10) used the derivative of $K_\pm^0(0)$ for $n = 2$, evaluated using the formula

$$K_+^{0'}(0) = \pm i\frac{2\beta - \eta_0 + \eta_0 b\sqrt{\beta}(d_0 + 2bd_1 + 3b^2 d_2)}{2\eta_0\beta}K_\pm(0) \tag{4.11}$$

Calculations have shown that the optimum version of the approximation is that with $n = 1$, assuming that the first and third conditions of (4.9) are satisfied. In that case, in order to determine d_0 and d_1, one has to set $d_2 = 0$ in the system of linear algebraic equations (4.10) and retain the first and third equations, from which d_0 and d_1 are then determined:

$$d_0 = (2b_1\varepsilon_2 - b_3)\Delta_0^{-1}, \quad d_1 = (b_3 - b_1)\Delta_0^{-1}, \quad \Delta_0 = 2\varepsilon_2 - 1 \tag{4.12}$$

The quantities d_0 and d_1 evaluated by formulae (4.12) for different values of Poisson's ratio ν, are listed below

ν	0.10	0.20	0.25	0.30	0.35	0.40	0.45
d_0	8.1762	7.4446	6.9315	6.3164	5.5968	4.7638	3.7701
d_1	49.5091	25.7310	16.8167	9.6998	4.2849	0.5234	-1.5457

For $\nu \in [0, 0.48]$ the approximation error in $|K(u)|$ is at most 5%, while for $\nu \in [0.48, 0.49]$ it is at most 6%. The increase in the error ν values close to 0.5 is due to the fact that at $\nu = 0.5$ ($\beta = 0$) neither $K'_\pm(0)$ nor $K''(0)$ exists, as is obvious from relations (2.2) and (4.1).

In special cases, an approximation of type (4.1) may be constructed, defining the constants d_k on the basis of other conditions and other characteristic points, e.g., on the basis of branch points, Rayleigh poles, etc.

5. THE APPROXIMATE SOLUTION OF INTEGRAL EQUATION (3.2)

To obtain a solution of Eq. (3.2) by formulae (3.23), we will use an approximation to $K(u)$ of the form (4.1) with $n = 1$, in which the coefficients d_0 and d_1 are defined by formulae (4.12) for any values of Poisson's ratio ν. Substituting $K_+^0(u)$ from (4.2) for $K_+(u)$ in (3.23) and evaluating the quadratures, we obtain a solution $\varphi_\pm^L(x, p)$ of the integral equation (3.2) for problem 1

$$\Delta^{-1}\varphi_\pm^L(x, p) = \sum_{k=1}^{5} c_{1k}(p)\varphi_{1k}(x, p) + \sum_{k=6}^{7} c_{1k}(p)\varphi_{1k}(\pm\Lambda x \pm 1, p) \tag{5.1}$$

$$\varphi_{11}(x, p) = k_{\pi 0}\int_\beta^\infty \frac{m(\xi)}{\xi}\exp(-\xi x)d\xi$$

$$\varphi_{12}(x, p) = k_{\pi 0}\int_\beta^\infty \frac{m(\xi)}{\xi}\exp(-\xi x)d\xi$$

$$\varphi_{13}(x, p) = -\frac{2}{\pi^2}\int_\beta^\infty \frac{m(\xi)}{\xi}\exp\left(-\frac{\xi}{\Lambda}\right)d\xi\int_\beta^\infty \frac{m(\eta)}{(\eta+\xi)}\exp(-\eta x)d\eta$$

$$\varphi_{14}(x, p) = -\frac{2}{\pi}\int_\beta^\infty \frac{l(\xi)}{\xi^2}\exp(-\xi|x|)d\xi$$

$$\varphi_{15}(x, p) = 1, \quad \varphi_{16}(x, p) = x, \quad \varphi_{17}(x, p) = |x|H(\pm x)$$

$$c_{11}(p) = (\varepsilon^L(p) - \eta_1\Lambda(1 + \Lambda\gamma'_- + 2\Lambda\gamma_p^0))\Lambda^{-1}, \quad c_{1k} = \eta_1\Lambda, \quad k = 2, 3, 4$$

$$c_{15}(p) = (\varepsilon^L(p) - \eta_1\Lambda)(\Lambda K(0))^{-1}, \quad c_{16}(p) = K^{-1}(0)\eta_1\Lambda$$

$$c_{17}(p) = -c_{14}(p)(\Lambda K(0))^{-1}$$

$$\gamma_p^0 = \frac{1}{\pi}\int_\beta^\infty \frac{m(\xi)}{\xi}\exp\left(-\frac{\xi}{\Lambda}\right)d\xi, \quad k_{\pi 0} = (\pi K_-(0))^{-1}$$

$$m(\xi) = \begin{cases} m_1(\xi), & 1 \le \xi < \infty, \quad m_1(\xi) = \psi_1(\xi)\exp\left[\frac{1}{2}(d_0\omega(\xi) - d_1\omega^2(\xi))\right] \\ m_2(\xi), & \beta \le \xi \le 1, \quad m_2(\xi) = \psi_1(\xi)\exp\left(-\frac{1}{2}\psi_2(\xi)\right)\cos\psi_3(\xi) \end{cases}$$

$$\psi_1(\xi) = \frac{\eta_0 - \xi}{\xi\sqrt{\xi - \beta}}, \quad \psi_2(\xi) = (1 + \beta - 2\xi)d_0 + ((\xi - \beta)^2 - 6(\xi - \beta)(1 - \xi) + (1 - \xi)^2)d_1,$$

$$\psi_3(\xi) = \sqrt{\xi - \beta}\sqrt{1 - \xi}(2(1 + \beta - 2\xi)d_1 + d_0), \quad \omega(\xi) = (\sqrt{\xi - \beta} - \sqrt{\xi - 1})^2$$

$$\gamma'_+ = iK'_+(0)/K_+(0)$$

The function $l(u)$ is given by formula (3.7).

In exactly the same way, the same kind of approximation is used to derive from (3.23) a solution of integral equation (3.2) for the case of problem 2

$$\Delta^{-1}\varphi^L_{\pm}(x,p) = \sum_{k=1}^{6} c_{2k}(p)\varphi_{2k}(x,p) \tag{5.2}$$

$$\varphi_{21}(x,p) = \varphi_{11}(x,p),\quad \varphi_{22}(x,p) = \varphi_{12}(x,p),\quad \varphi_{23}(x,p) = k_{\pi 0}\int_{\beta}^{\infty}\frac{m(\xi)}{\xi}\exp(-\xi x)d\xi$$

$$\varphi_{24}(x,p) = 1,\quad \varphi_{25}(x,p) = x,\quad \varphi_{26}(x,p) = x^2$$

$$c_{21}(p) = -(\varepsilon^L(p) - \eta_2\Lambda(1+2\Lambda\gamma'_- + 2(2\gamma'^2_- + \gamma''_-)))\Lambda^{-1}$$

$$c_{22}(p) = 2\eta_2\Lambda(1+2\Lambda\gamma'_-),\quad c_{23}(p) = 2\eta_2\Lambda^2$$

$$c_{24}(p) = -K^{-1}(0)(c_{21}(p) + \gamma'_+ c_{22}(p) + (\gamma'^2_+ + \gamma''_+/2)c_{23}(p)$$

$$c_{25}(p) = -K^{-1}(0)(c_{22}(p) - c_{23}(p))\gamma'_+,\quad c_{26}(p) = -\frac{1}{2}K^{-1}(0)c_{23}(p)$$

$$\gamma''_+ = K''_+(0)/K_+(0)$$

The quantities γ'_+, $k_{\pi 0}$, $m(\xi)$ and $\varphi^L_{1k}(x,p)$ are given by formulae (5.1), and γ'_- and γ''_- are given by formulae (3.18) and (3.19).

Note that the solutions of Eq. (3.2) have been obtained in the class of integrable functions that permit singularities of the Laplace transform of the contact stresses at the boundary of the contact area at points $x = \pm 1$, that is, $\varphi^L_{\pm}(x,p) = \omega(x,p)(1 \pm x)^{-1/2}$, where $\omega(x,p) \in C_{[-1,1]}$. This feature of relations (5.1) and (5.2) is conveyed by the function $\varphi^L_{11}(x,p)$ after evaluating the quadrature. In addition, $\varphi^L_{\pm}(x,p)$ in (5.1) has a logarithmic singularity as $x \to 0$, which is conveyed by its function $\varphi^L_{14}(x,p)$.

6. THE SOLUTION OF PROBLEMS 1 AND 2

An asymptotic solution of the transient dynamic contact problems formulated in Section 1, for wedge-shaped and paraboloid punches penetrating an elastic half-space, is obtained by reverting to the original functions from their Laplace transforms, obtained above as asymptotic solutions of Eqs (3.2) and (3.3), in which $\varphi^L_{\pm}(1 \pm x/\Lambda, p)$ and $\varphi_\infty(x/\Lambda, p)$ are given by formulae (5.1) for problem 1 and by formulae (5.2) for problem 2. As a result, the solution of problems 1 and 2 is given by the formula

$$\varphi(x,t) = \varphi_+\left(\frac{a(1+x)}{c_2}, t\right) + \varphi_-\left(\frac{a(1-x)}{c_2}, t\right) - \varphi_\infty\left(\frac{ax}{c_2}, t\right) \tag{6.1}$$

in which $\varphi_{\pm}(x,t)$ and $\varphi_\infty(x,t)$ for problem 1 are given by the formulae

$$\Lambda^{-1}\varphi_{\pm}(u,t) = \sum_{k=1}^{2} H(t-\beta u)u^{-3/2+k}\Phi_{1k}(u,t) + H(t-\beta u - t_1)u^{1/2}\Phi_{13}(u,t) +$$

$$+ H(t - |\beta u - t_1|)\Phi_{14}(|u - t_2|, t) + \sum_{k=5}^{6} H(t)\Phi_{1k}(u,t) + H(t)\Phi_{17}(|u - t_2|, t) \tag{6.2}$$

$$\Delta^{-1}\varphi_\infty(u,t) = t_2 K^{-1}(0)(\dot\varepsilon(t) + \varepsilon(0)) - \theta_1 c_2 H(t-\beta u)\Phi_{14}(u,t) - \theta_1 c_2 K^{-1}(0)u\delta(t) \tag{6.3}$$

$$\Phi_{1k}(u,t) = k_{\pi 0}\int_{\beta u}^{t-\beta u} c_{1k}(\tau)q_k(t-\tau, u)m_*((t-\tau)u^{-1})d\tau,\quad k = 1, 2$$

$$\Phi_{13}(u,t) = -\frac{2}{\pi^2}\int_{\beta u+t_1}^{t}\int_{\beta}^{(\tau-\beta u)/t_2} c_{13}(\tau)q_1(\tau-\xi t_2,u)m_*((\tau-\xi t_2)u^{-1})m(\xi)\xi^{-1}d\xi d\tau$$

$$\Phi_{14}(u,t) = -\frac{2}{\pi}\int_{\beta}^{t/u}\xi^{-2}l(\xi)d\xi, \quad \Phi_{15}(u,t) = 1, \quad \Phi_{16}(u,t) = \Phi_{17}(u,t) = u$$

$$c_{11}(t) = -t_2(\dot{\varepsilon}(t)+\varepsilon(0))+\eta_1 t_2^{-1}\left(a\delta(t)+\gamma'_-H(t)+2t_2^{-1}H(t-t_1)\int_{\beta}^{t/t_2}\xi^{-1}m(\xi)d\xi\right)$$

$$c_{1k}(t) = \eta_1 t_2^{-1}H(t), \quad k = 2,3,4$$

$$c_{15}(t) = t_2K^{-1}(0)(\dot{\varepsilon}(t)+\varepsilon(0))-\eta_1K^{-1}(0)\delta(t)$$

$$c_{16}(t) = \eta_1 t_2^{-1}K^{-1}(0)\delta(t), \quad c_{17}(t) = -\eta_1K^{-1}(0)\delta(t)$$

where

$$m_*(\xi) = \frac{\xi\sqrt{\xi-\beta}}{\eta_0-\xi}m(\xi), \quad q_k(t,u) = \frac{\eta_0 u-t}{t^k\sqrt{t-\beta u}}, \quad t_k = \frac{a}{c_k}, \quad k = 1,2$$

The functions $m(\xi)$ and $l(\xi)$ were defined after formulae (5.1) and (3.7), respectively. For problem 2, $\varphi_\pm(x,t)$ and $\varphi_\infty(t)$ take the form

$$\Delta^{-1}\varphi_\pm(u,t) = \sum_{k=1}^{3}H(t-u\beta)u^{-3/2+k}\Phi_{2k}(u,t)+\sum_{k=4}^{6}H(t)c_{2k}(t)\Phi_{2k}(u) \tag{6.4}$$

$$\Delta^{-1}\varphi_\infty(u,t) = t_2K^{-1}(0)(\dot{\varepsilon}(t)+\varepsilon(0))-\theta_2t_2a^2u^2\delta(t)-\theta_2K''(0)ac_2t) \tag{6.5}$$

$$\Phi_{2k}(u_1,t) = k_{\pi 0}\int_{\beta u}^{t-\beta u}c_{2k}(\tau)q_k(t-\tau,u)m_*((t-\tau)u^{-1})d\tau, \quad k = 1,2,3$$

$$\Phi_{24}(u) = 1, \quad \Phi_{25}(u) = u, \quad \Phi_{26}(u) = u^2$$

$$c_{21}(t) = -t_2(\dot{\varepsilon}(t)+\varepsilon(0))+\theta_2\Lambda(a^2\delta(t)+2a\gamma'_-+c_2t(2\gamma'^2_-+\gamma''_-))$$

$$c_{22}(t) = 2\theta_2a^2(1+2t_2\gamma'_-t), \quad c_{23}(t) = 2\theta_2c_2at$$

$$c_{24}(t) = -K^{-1}(0)(c_{21}(t)+\gamma'_+c_{22}(t)+(\gamma'^2_++\gamma''_+/2)c_{23}(t))$$

$$c_{25}(t) = -K^{-1}(0)(\dot{c}_{22}(t)-\dot{c}_{23}(t))\gamma'_+$$

$$c_{26}(t) = -K^{-1}(0)\ddot{c}_{23}(t)/2$$

In formulae (6.2)–(6.5), as in all previous formulae, the parameter a – the half-width of the contact area of the punch with the elastic medium – is a function of the time t, even when it occurs in integrals with respect to τ. Formulae (6.2)–(6.5) for $\varphi_\pm(u,t)$ and $\varphi_\infty(u,t)$ cannot yet be solutions of problems 1 and 2, since they involve singularities of the root type $(1\pm x)^{-1/2}$ at the boundaries of the contact area, indicating the existence of sources of elastic energy (or discontinuities in the surface of the elastic medium) at the boundaries of the contact area. To obtain smooth solutions, bounded at the ends of the contact area, the coefficients of the terms $(1\pm x)^{-1/2}$ must vanish as $x\to\pm 1$, that is, the following conditions must hold

$$\lim_{x \to \pm 1} \Phi_{11}\left(\frac{a(1 \pm x)}{c_2}, t\right) = 0 \quad \text{for problem 1} \tag{6.6}$$

$$\lim_{x \to \pm 1} \Phi_{21}\left(\frac{a(1 \pm x)}{c_2}, t\right) = 0 \quad \text{for problem 2} \tag{6.7}$$

Satisfaction of these conditions yields the following equations defining the half-width $a(t)$ of the contact area

$$a(t) = \frac{\sqrt{t}}{\theta_1}\int_0^t \frac{\dot{\varepsilon}(\tau)}{\sqrt{t-\tau}}d\tau - 2c_2\gamma'_- t - \frac{4}{\pi}c_2 K_-(0)H(t-t_1)\sqrt{t}\int_\beta^{t/t_2} m(\xi)\frac{\sqrt{t-\xi t_2}}{\xi}d\xi, \quad t_k = \frac{a}{c_k}, \tag{6.8}$$

$k = 1, 2$ for problem 1

$$a(t) = -2\gamma'_- c_2 t + \left(\frac{\sqrt{t}}{\theta_2}\int_0^t \frac{\dot{\varepsilon}(\tau)}{\sqrt{t-\tau}}d\tau - \frac{8}{3}(c_2 t)^2\left(\frac{1}{2}\gamma'^2_- + \gamma''_-\right)\right)^{1/2} \quad \text{for problem 2} \tag{6.9}$$

Equations (6.8) and (6.9) are written in a form convenient to be run on a computer. The constants γ'_-, γ''_- and $\chi_- = \gamma'^2_-/2 + \gamma''_-$ are given below.

ν	0.10	0.20	0.25	0.30	0.35	0.40	0.45
γ'_-	0.3519	0.3632	0.3600	0.3434	0.2996	0.1894	−0.1502
γ''_-	−0.4169	−0.4344	−0.3814	−0.2244	0.1747	1.2602	5.3802
χ_-	−0.3550	−0.3684	−0.3166	−0.1654	0.2196	1.2781	5.3915

It can be seen that the parameters γ'_-, γ''_- and χ_- are significantly influenced by Poisson's ratio ν, and consequently the width of the contact area depends very much on the properties of the elastic medium. At small t values one has the following estimates of $a(t)$

$$a(t) = 2(\theta_1^{-1}v_0 - c_2\gamma'_-)t + O(t) \quad \text{as} \quad t \to 0 \quad \text{for problem 1} \tag{6.10}$$

$$a(t) = \sqrt{2\theta_2^{-1}v_0}\sqrt{t} + O(t) \quad \text{as} \quad t \to 0 \quad \text{for problem 2} \tag{6.11}$$

It follows from (6.10) that a necessary condition for the requirement $a(t) \geq 0$ to hold is the inequality

$$\theta_1^{-1}v_0 - c_2\gamma'_- \geq 0, \quad \theta_1 = \operatorname{ctg}\alpha$$

which implies the validity of the restrictions imposed on the angle 2α of the wedge

$$2\operatorname{arctg}(c_2\gamma'_-/v_0) \leq 2\alpha \leq \pi \tag{6.12}$$

Formulae (6.10) also implies an estimate for $\dot{a}(t)$, the rate of change in the half-width of the contact area

$$\dot{a}(t) = 2(\theta_1^{-1}v_0 - c_2\gamma'_-) + O(\sqrt{t}), \quad t \to 0 \tag{6.13}$$

It follows from (6.11) that at small t the contact area exists for fairly small opening spans θ_2 of the parabola, and its rate of expansion $\dot{a}(t)$ admits of the estimate

$$\dot{a}(t) = \sqrt{\frac{v_0}{2\theta_2}}\frac{1}{\sqrt{t}} + O(1), \quad t \to 0 \tag{6.14}$$

whence it follows that at the initial time the rate of expansion of the contact area under a parabolic punch is unbounded [5].

To conclude this section, we must note that the solution of problem 1 contains a logarithmic singularity with respect to the x coordinate as $x \to 0$, generated by the corner point of the wedge and contained in (6.2) and (6.3) in the function $\Phi_{14}\left(\frac{|x|a}{c_2}, t\right)$.

Isolation of the singularity in $\Phi_{14}(u, t)$ brings it to the form

$$\Phi_{14}\left(\frac{|x|a}{c_2}, t\right) = \frac{2}{\pi}\left[\ln\left|\frac{c_1 t + \sqrt{(c_1 t)^2}}{|x|a}\right| H\left(t - \frac{|x|a}{c_1}\right) + \Phi^*_{14}\left(\frac{|x|a}{c_2}, t\right)\right]$$

$$\Phi^*_{14}(u, t) = \left(\int_\beta^{t/u} \frac{d\xi}{(\xi^2 - \beta^2)^{1/2}} + \int_\beta^{t/u} g(\xi)d\xi\right) H(t - \beta u)$$

$$g(u) = \begin{cases} g_1(u) & 1 \le u < \infty, \quad g_1(u) = u^{-2}(l_1(u) - 4u^2\sigma_{10}\sigma_{20}l_0^{-1}(u))\sigma_{20} + 1 \\ g_2(u) & \beta \le u \le 1, \quad g_2(u) = u^{-2}l_1(u)\sigma_{20} \end{cases}$$

Expressions for $l_k(u)$ ($k = 0, 1$) are contained in (3.7).

If condition (6.9) holds, the solution (6.1), (6.4), (6.5) for problem 2 does not contain singularities.

7. THE MOTION OF PUNCHES ON THE ELASTIC MEDIUM

The penetration of the punches into the elastic half-space is treated as the motion of absolutely rigid bodies and reduces to determining the motion of their centres of mass, which are situated on the axes of symmetry of the punches, that is, the y axis. In that case, the motion of the punch may be considered as the motion of a point mass of mass m. The differential equation of motion of the punch with initial conditions is [1, 2]

$$m\ddot{\varepsilon}(t) = Q(t), \quad \dot{\varepsilon}(0) = v_0, \quad \varepsilon(0) = \varepsilon_0 \tag{7.1}$$

where $Q(t)$, the force of elastic resistance of the medium to the penetration of the punch, is related to the force of contact action $P(t)$ by the formula

$$Q(t) = \int_{-a}^{a} \sigma_{yy}(x, 0, t)dx = -\int_{-a}^{a} \varphi(x, t)dx = -P(t), \quad P(t) = \int_{-a}^{a} \varphi(x, t)dx \tag{7.2}$$

where $\varphi(x, t)$ are the contact stresses, given by formulae (6.1) for the problems is question.

Thus, formula (7.2) can be used to determine $Q(t)$. We shall now outline a briefer way to determine $P(t)$. to determine $P(t)$, one first finds $P^L(p)$

$$P^L(p) = \int_{-a}^{a} \varphi^L(x, p)dx \tag{7.3}$$

To do this, consider integral equation (1.5). Suppose a solution of that equation is known when the right-hand side has the form $f^L(x, p) = 1$. Denote it by $\varphi_0^L(x, p)$. Multiply the left- and right-hand sides of Eq. (1.5) by $a\varphi_0^L(x, p)dx$ and integrate the resulting equality with respect to x from -1 to 1. Inverting the order of integration and using the evenness of $K(u)$ and the properties of $\varphi_0^L(x, p)$, we obtain the formulae

$$P^L(p) = a\lambda_0 \int_{-1}^{1} \varphi_0^L(x, p)dx + a\lambda_1 \int_{-1}^{1} \varphi_0^L(x, p)|x|dx \quad \text{for problem 1} \tag{7.4}$$

$$P^L(p) = a\lambda_0 \int_{-1}^{1} \varphi_0^L(x, p)dx + a\lambda_2 \int_{-1}^{1} \varphi_0^L(x, p)x^2dx \quad \text{for problem 2} \tag{7.5}$$

Expression for λ_0, λ_1, λ_2 are given in (3.5) and (3.6).

The aforementioned solution $\varphi_0^L(x,p)$ of Eq. (1.5) for the case when $f^L(x,p) = 1$ is found by elementary means [1, 2] as the superposition (3.1), in which

$$\varphi_+^L(x, p) = \frac{1}{\pi\Lambda K_-(0)}\left[-\int_\beta^\infty m(u)e^{-ux}du + \frac{\pi}{K_+(0)}\right]$$
$$\varphi_\infty^L(x, p) = \frac{1}{\Lambda K(0)} \tag{7.6}$$

The function $m(u)$ was defined in formulae (5.1). Substituting $\varphi_0^L(x, p)$ into relations (7.4) and (7.5) and reverting from the Laplace transform to the original function, we obtain

$$a^{-1}P^L(p) = (\lambda_0+\lambda_1)P_1^L(p) - \lambda_1\Lambda P_{2+}^L(p) + 2\lambda_1\Lambda P_{20}^L(p) + (2\lambda_0+\lambda_1)\Lambda^{-1}K^{-1}(0) \tag{7.7}$$

for problem 1

$$a^{-1}P^L(p) = (\lambda_0+\lambda_1)P_1^L(p) - 2\lambda_1\Lambda P_2^L(p) + 2\lambda_1\Lambda^2 P_3^L(p) + \left(2\lambda_0 + \frac{2}{3}\lambda_1\right)\Lambda^{-1}K^{-1}(0) \tag{7.8}$$

for problem 2

where

$$P_k^L(p) = -\frac{2}{\pi K_-(0)}\int_\beta^\infty \frac{m(\xi)}{\xi^2}\left(1 - \exp\left(-\xi\frac{2}{\Lambda}\right)\right)d\xi, \quad k = 1, 2, 3$$

$$P_{2+}^L(p) = -\frac{2}{\pi K_-(0)}\int_\beta^\infty \frac{m(\xi)}{\xi^2}\left(1 + \exp\left(-\xi\frac{2}{\Lambda}\right)\right)d\xi$$

$$P_{20}^L(p) = -\frac{2}{\pi K_-(0)}\int_\beta^\infty \frac{m(\xi)}{\xi^2}\exp\left(-\frac{\xi}{\Lambda}\right)d\xi$$

The function $m(u)$ is given in (5.1) and λ_0 and λ_1 are given in (3.6).

Reverting to the originals from $P^L(p)$ in relations (7.7) and (7.8), we obtain

$$(a\Delta)^{-1}P(t) = 2t_2K^{-1}(0)(\dot{\varepsilon}(0) + \varepsilon(0)) +$$
$$+ P_{1\varepsilon}(t) - \theta_1c_2(t_2P_1(t) - P_{2+}(t) + 2P_{20}(t) + t_2^2K^{-1}(0)\delta(t)) \quad \text{for problem 1} \tag{7.9}$$

$$(a\Delta)^{-1}P(t) = 2t_2K^{-1}(0)(\dot{\varepsilon}(t) + \varepsilon(0)) +$$
$$+ P_{1\varepsilon}(t) - \theta_2c_2\left(t_2^2P_1(t) - 2t_2P_2(t) + P_3(t) + \frac{2}{3}t_2^3K^{-1}(0)\delta(t)\right) \quad \text{for problem 2} \tag{7.10}$$

where

$$P_k(t) = -\frac{2}{\pi K_-(0)}\int_\beta^\infty \frac{m(\xi)}{\xi^k}(t^{k-1}H(t) - (t-2\xi t_2)^{k-1}H(t-2\xi t_2))d\xi, \quad k = 1, 2, 3$$

$$P_{1\varepsilon}(t) = -\frac{2}{\pi K_-(0)}\int_\beta^\infty \frac{m(\xi)}{\xi}(\varepsilon(t) - \varepsilon(t-2\xi t_2)H(t-2\xi t_2))d\xi,$$

$$P_{2+}(t) = -\frac{2}{\pi K_-(0)}\int_\beta^\infty \frac{m(\xi)}{\xi}(tH(t) + (t-2\xi t_2)H(t-2\xi t_2))d\xi$$

$$P_{20}(t) = -\frac{2}{\pi K_{-}(0)}\int_{\beta}^{\infty}\frac{m(\xi)}{\xi}(t-\xi t_2)H(t-\xi t_2)d\xi$$

$t_k - a/c_k$ ($k = 1, 2$). Δ is as in (1.6), and $\delta(t)$ and $H(t)$ are the Dirac delta function and the Heaviside function, respectively.

To determine the signs of the motion of the punches $\varepsilon(t)$ in the solution of problems 1 and 2, formulae (7.9) and (7.10) just found for the function $P(t)$ are substituted into the right-hand side of the ordinary differential equation (7.1) taking formulae (7.2) into consideration. The Volterra-type integrodifferential equation for the unknown function $\varepsilon(t)$ thus obtained for each of the problems contains one more unknown, $a(t)$, which can be determined using the previously obtained additional conditions (6.8) and (6.9) for problems 1 and 2, respectively.

In a numerical solution of the integrodifferential equations (7.1), (7.2), (7.9) and (7.10), at each step of the integration (the determination of $\varepsilon(t)$) the value of the other unknown function $a(t)$ corresponding to that particular instant of time is determined numerically as the root of algebraic equation (6.8) or (6.9) for problems 1 and 2, respectively, taking care at the same time to satisfy the natural initial condition $a(0) = 0$. This algorithm for solving the problem is fairly easy to implement in a MathCad environment.

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